Does deconvolution exist?

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Abstract

Blurring of a photographic image by a wrong focus can be modeled by convolution. This paper discusses some points for the inverse operation with particular interest on the set of integers $\mathbb{Z}$.

MSC2010: 65R30 (Improperly posed problems), 94A08 (Image processing).

1 Introduction.

Briefly a problem is *ill-posed* if there is a "bad" transformation A ("signal" $\mapsto$ "blurred signal" for example) and one tries recovering the preimage of any y, expecting to find an x such that $Ax = y$. Difficulties could lie in: A is not one-to-one, or very different initial points may have very closed image (see Section 4), or (frequently this happens simultaneously) the map A is not onto.

Photographic images often present blurring, for example due to a wrong focus setting. Several other defects due to different causes are possible (cf. [Be]). Defect of focus is roughly equivalent to convolution of the image source with the brightness of the image of one point light in 0[1]. Numerous papers use the word *deconvolution*. Is it more than a word? Surely this belongs to the class of ill-posed problems (see [TA, Ch.IV pp.91–115]).

Several authors add stochastic component. There is a clear reason: when the map "signal" $\mapsto$ "blurred signal" is not onto (this may highly depends on the functional space under consideration), finding a preimage to any point

[1] The density could be $k\mathbf{1}_{B(0,r)}$ where $r > 0$ is a radius and $k = (\pi r^2)^{-1}$. Question in dimension 2 and with the Euclidean norm: is it a zero divisor for convolution? We will see (Section 2) that in dimension 1 we do have a zero divisor.

in the target space of A needs some stochastic adjustment. Many papers speak of *bayesian*. The signal itself can be seen as a trajectory of a stochastic process[2]. *Robustness* may also be refered to because the perturbation is not exactly known. A lot of recent papers use *wavelets*[3].

Literature is prolific and difficult to understand. The words *mask*, *sharpening* are keys on the Net, as also *filtration* and *denoising*. An astonishing algorithm is due to P.H. van Cittert: see Wikipedia (German) [J, VC]. A paper with an heralding title, which quotes van Cittert, and illustrate the interest to the question outside of the purely mathematical world is [Bi].

We will give some calculi with the space $\mathbb{Z}$ (dimension 1) and refind the threshhold $\frac{1}{2}$ highlighted by C. Duval [D] (see Section 6).

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2 Convolution. Notations. Zero divisors.

Convolution and Fourier transform have as framework $\mathbb{R}^d$ (which is its own dual group), or $\mathbb{Z}^d$ and its dual group[4] $\mathbb{T}^d$ (maybe the groups $\mathbb{Z}/n\mathbb{Z}$?). When μ and ν are bounded measures on $\mathbb{R}^d$, their convolution product, denoted by $\mu * \nu$, is the image (pushforward[5]) by $(x, y) \mapsto x + y$ of their product $\mu \otimes \nu$ (cf. the sum of independant random variables in Probability). For Lebesgue integrable functions on $\mathbb{R}^d$, their convolution product is classically

$$f * g : x \mapsto \int_{\mathbb{R}^d} f(x - y)\, g(y)\, dy\,.$$

The excellent paper by K.A. Ross [R] examines mainly convolution of L^1 functions.

Concerning convolution of distributions, L. Schwartz begins by the case of two when one of them has compact support. Then he proves [S, vol.2, Th.VII p.14] that the convolution of a finite number of distributions which have all, except one, compact supports is associative et commutative and then [S, vol.2, ch.6 §5 p.26] moves to more general situations. He proves that, in dimension $d = 1$, the *convolution algebra* $\mathcal{D}'_+$ (the set of distributions with supports limited on left) has no zero divisors [S, vol.2, ch.6 Th.XIV p.29].

[2] Cf. the Wiener filter, I learned in R. Pallu de La Barrière [PB].
[3] Using the Fourier transform is tempting but disappointing.
[4] Cf. Fourier series.
[5] Cf. the writing $\mu * \nu = S_\#(\mu \otimes \nu)$ where S denotes the sum.

The same result holds for $\mathcal{D}'_-$ (the set of distributions with supports limited on right).

The unit mass in 0, δ_0, is always a neutral element and convolution by δ_x ($x \in \mathbb{R}^d$) amounts to translation by the vector x. Uniqueness of a possible inverse holds when one works in a subspace where associativity holds (see (1) hereafter). We will see cases where several inverses do coexist (Theorem 1).

Let us show that δ_x has as unique inverse δ_{-x} (who doubts it?). Suppose H is another distribution inverse of δ_x. Among the three distributions δ_x, δ_{-x} and H, two have compact support, hence associativity holds and

$$H = H * \delta_0 = H * (\delta_x * \delta_{-x}) = (H * \delta_x) * \delta_{-x} = \delta_0 * \delta_{-x} = \delta_{-x}\,. \tag{1}$$

When dealing with $\mathbb{Z}$ and measures such as $\mu = \sum_{n\in\mathbb{Z}} x_n\, \delta_n$, or $\nu = \sum_{n\in\mathbb{Z}} y_n\, \delta_n$, the point of view of convolution is to consider the function $n \mapsto x_n$ defined on $\mathbb{Z}$ (resp. $n \mapsto y_n$). The convolution of μ and ν returns to the convolution $z := x*y$ where $z_n = \sum_{k\in\mathbb{Z}} x_{n-k}\, y_k$. Next h will equivalently denote a measure or a function on $\mathbb{Z}$.

Examples of zero divisors.

1) With $\mathbb{R}$ let consider the *gate function* $h = \mathbf{1}_{[-1,1]}$. Then

$$h * \mathbf{1}_{\cup_{n\in\mathbb{Z}}[2n,2n+1]} = h * \left(\frac{1}{2}\mathbf{1}_{\mathbb{R}}\right)$$

hence $f \mapsto h * f$ is not injective on $L^\infty(\mathbb{R})$, and one has a zero divisor:

$$h * \left(\mathbf{1}_{\cup_{n\in\mathbb{Z}}[2n,2n+1]} - \frac{1}{2}\mathbf{1}_{\mathbb{R}}\right) = 0\,.$$

2) With $\mathbb{Z}$, take $h := \frac{1}{2}\,\delta_0 + \frac{1}{2}\,\delta_1$ or $h := \frac{1}{4}\,\delta_{-1} + \frac{1}{2}\,\delta_0 + \frac{1}{4}\,\delta_1$. Then there holds

$$h * \mathbf{1}_{2\mathbb{Z}} = h * \left(\frac{1}{2}\mathbf{1}_{\mathbb{Z}}\right), \tag{2}$$

hence $f \mapsto h * f$ is not injective on $\ell^\infty(\mathbb{Z})$, and one has the zero divisor:

$$h * \left(\mathbf{1}_{2\mathbb{Z}} - \frac{1}{2}\mathbf{1}_{\mathbb{Z}}\right) = 0\,.$$

3 Convolution and inverse, particular cases.

Let us begin by $h = a\,\delta_0 + (1-a)\,\delta_1$ ($a \in]0,1[$) (a kind of "gate function").

Lemma 1 *Let $a \in]0,1[$ and $h = a\,\delta_0 + (1-a)\,\delta_1$. Then an inverse of h in $\mathcal{D}'_+(\mathbb{R})$ is*[6]

$$J = \frac{1}{a}\,\delta_0 - \frac{1-a}{a^2}\,\delta_1 + \frac{(1-a)^2}{a^3}\,\delta_2 + \dots \tag{3}$$

(the limit is for the weak topology $\sigma(\mathcal{D}', \mathcal{D})$)

PROOF. Indeed

$$h * \frac{1}{a}\sum_{n=0}^{k}\Big[-\frac{1-a}{a}\Big]^n \delta_n = \delta_0 + (-1)^k \Big[\frac{1-a}{a}\Big]^{k+1} \delta_{k+1}$$
$$\to \delta_0$$

because for any α_n, $\alpha_n\,\delta_n \to 0$ in the topology $\sigma(\mathcal{D}', \mathcal{D})$ when $n \to \infty$. □

Lemma 2 *An inverse of h in $\mathcal{D}'_-(\mathbb{R})$ is*

$$\frac{1}{1-a}\,\delta_{-1} - \frac{a}{(1-a)^2}\,\delta_{-2} + \frac{a^2}{(1-a)^3}\,\delta_{-3} - \frac{a^3}{(1-a)^4}\,\delta_{-4} + \dots \tag{4}$$

(the limit still for $\sigma(\mathcal{D}', \mathcal{D})$)

PROOF. One can write

$$h = (1-a)\,\delta_1 * (\delta_0 + \frac{a}{1-a}\,\delta_{-1})\,.$$

Then $(1-a)\,\delta_1$ admits the inverse $\frac{1}{1-a}\,\delta_{-1}$ and for the second factor one can develop "on left" as in the preceding lemma. □

Theorem 1 *The distribution $\frac{1}{2}\,(\delta_0 + \delta_1)$ on $\mathbb{R}$ admits several inverses in $\mathcal{D}'$ with respect to convolution (the limits are for $\sigma(\mathcal{D}', \mathcal{D})$):*

$$J_1 = 2\,\lim_{k\to\infty}\sum_{n=0}^{k}(-1)^n \delta_n = 2\,(\delta_0 - \delta_1 + \delta_2 - \delta_3 + \dots)\,, \tag{5}$$

[6] Cf. the known formula $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$ for $x \in \mathbb{R}$.

$$J_2 = 2 \lim_{k\to\infty} \sum_{n=1}^{k} (-1)^{n-1} \delta_{-n} = 2\,(\delta_{-1} - \delta_{-2} + \delta_{-3} - \delta_{-4} + \ldots)$$

and specially $H = \frac{1}{2}\,(J_1 + J_2)$ *i.e.*

(6) $$H = \ldots - \delta_{-4} + \delta_{-3} - \delta_{-2} + \delta_{-1} + \delta_0 - \delta_1 + \delta_2 - \delta_3 + \ldots$$

Moreover for any $f \in \mathbb{R}^{(\mathbb{Z})}$ *(the space of real sequences on* $\mathbb{Z}$ *with compact supports)*

$$(f * h) * H = f\,.$$

REMARKS. For any $\lambda \in \mathbb{R}$, $\lambda\, J_1 + (1-\lambda)\, J_2$ is also an inverse of h. And $J_1 - J_2$ forms with h a couple of zero divisors.

PROOF. The lemmas imply the assertions about inverses. The last formula follows from the fact that f and h have compact supports, hence $(f*h)*H = f*(h*H) = f*\delta_0$. □

Now we turn to a measure carried by $\{-1,0,1\}$, still positive with total mass 1. With the parameter $a \in \left]\frac{1}{2}, 1\right[$

(7) $$h := \frac{1-a}{2}\,\delta_{-1} + a\,\delta_0 + \frac{1-a}{2}\,\delta_1$$

or, with the parameter $b = \frac{1-a}{2} \in \left]0, \frac{1}{4}\right[$ which will be often better suited,

$$h := b\,\delta_{-1} + (1-2b)\,\delta_0 + b\,\delta_1\,.$$

Lemma 3 *Let* $b \in \left]0, \frac{1}{4}\right[$. *Then*

$$\lambda = \frac{1}{2b}\left[2b - 1 + \sqrt{1-4b}\,\right]$$

belongs to $]-1,0[$, *tends to* 0 *if* $b \to 0$, *and tends to* -1 *if* $b \to 1/4$.

PROOF. Elementarily λ is a root of the equation $\lambda^2 + \frac{1-2b}{b}\,\lambda + 1 = 0$. One has $\lambda \le 0$ because

$$\begin{aligned} 2\,b - 1 + \sqrt{1-4b} \le 0 &\iff \sqrt{1-4b} \le 1 - 2\,b \\ &\iff 1 - 4b \le (1-2b)^2 \\ &\iff 1 - 4b \le 1 - 4b + 4b^2 \end{aligned}$$

and $\lambda > -1$ because

$$2b - 1 + \sqrt{1-4b} > -2b \iff \sqrt{1-4b} > 1 - 4b$$

which holds, since on $]0,1[$, $\sqrt{x}$ is $> x$. The convergences are easy. □

Theorem 2 *Let h given by (7)*

$$h := \frac{1-a}{2}\,\delta_{-1} + a\,\delta_0 + \frac{1-a}{2}\,\delta_1\,.$$

Let c defined by

$$\forall n \in \mathbb{Z}, c_n = \lambda^{|n|}\,\left(\sqrt{1-4b}\right)^{-1}\,.$$

The c_n are alternatively > 0 and < 0 and $\sum_{n\in\mathbb{Z}} c_n = 1$. The measure (or sequence) c is an inverse of h, that is $h{*}c = \delta_0$. Moreover for any $f \in \ell^\infty(\mathbb{Z})$

$$(f * h) * c = f\,.$$

Remark. Since $-1 < \lambda < 0$, c considered as a function oscillates as the famous *cardinal sine function*: $\operatorname{sinc} x = \frac{\sin x}{x}$ (cf. also the *mexican hat*). This seems quite general. For another comment see Section 6.

Proof. 1) One has

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} c_n &= c_0 + 2\sum_{n\geq 1} c_n \\
&= c_0\left(1 + 2\sum_{n\geq 1}\lambda^n\right) \\
&= c_0\left(1 + 2\,\frac{\lambda}{1-\lambda}\right) \\
&= \frac{1}{\sqrt{1-4b}}\,\frac{1+\lambda}{1-\lambda} \\
&= \frac{1}{\sqrt{1-4b}}\,\frac{4b-1+\sqrt{1-4b}}{1-\sqrt{1-4b}} \\
&= 1\,.
\end{aligned}$$

2) Firstly

$$(h*c)_n = \sum_{i\in\mathbb{Z}} h(n-i)\,c_i = \sum_{i\in\mathbb{Z}} h(i)\,c_{n-i}\,.$$

For $n = 0$ this gives

$$\begin{aligned}(h * c)_0 &= h(-1)\, c_1 + h(0)\, c_0 + h(1)\, c_{-1} \\ &= b\,\frac{\lambda}{\sqrt{1-4b}} + (1-2b)\,\frac{1}{\sqrt{1-4b}} + b\,\frac{\lambda}{\sqrt{1-4b}} \\ &= \frac{1}{\sqrt{1-4b}}\,[2b\,\lambda + 1 - 2\,b] \\ &= \frac{1}{\sqrt{1-4b}}\left[2b - 1 + \sqrt{1-4b} + 1 - 2\,b\right] \\ &= 1\,.\end{aligned}$$

For $n \geq 1$ this gives

$$\begin{aligned}(h * c)_0 &= h(-1)\, c_{n+1} + h(0)\, c_n + h(1)\, c_{n-1} \\ &= b\,(c_{n-1} + c_{n+1}) + (1-2b)\, c_n \\ &= c_0\,[b\,\lambda^{n-1} + b\,\lambda^{n+1} + (1-2b)\,\lambda^n] \\ &= \frac{\lambda^{n-1}}{\sqrt{1-4b}}\,[b + (1-2b)\lambda + b\lambda^2] \\ &= 0\end{aligned}$$

because $\lambda^2 + \frac{1-2b}{b}\,\lambda + 1 = 0$.

3) As for $(f * h) * c$, the functions are respectively, bounded for f, with compact support for h, integrable for c (convergent sum). So associativity holds. $\square$

4 Illustration (pictures on $\mathbb{Z}$).

A monochrome photographic image can be modelized by a (measurable) function $f : \mathbb{R}^2 \to [0, 1]$, f measuring the brightness.

We will expose some examples with $f : \mathbb{Z} \to [0, 1]$, that is a one dimensional picture formed from pixels. So the basic space is $\ell^\infty(\mathbb{Z})$. Another natural space is $\mathbb{R}^{(\mathbb{Z})}$ that is the space of real sequences on $\mathbb{Z}$ with compact supports (this is Bourbaki's notation); it is a natural space since pictures do have compact supports. Other vector spaces could be considered in abstract studies ($p \in\,]1, \infty[$):

$$\mathbb{R}^{(\mathbb{Z})} \subset \ell^1(\mathbb{Z}) \subset \ell^p(\mathbb{Z}) \subset c_0(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z}) \subset \mathbb{R}^{\mathbb{Z}}\,.$$

As measures spaces, $\mathbb{R}^{\mathbb{Z}} \sim \mathcal{M}(\mathbb{Z})$ (the space of all measures on $\mathbb{Z}$) and $\ell^1(\mathbb{Z}) \sim \mathcal{M}^b(\mathbb{Z})$ (the space of all bounded measures on $\mathbb{Z}$).

Let us consider the linear map

$$A := \begin{cases} \mathbb{R}^{\mathbb{Z}} \longrightarrow \mathbb{R}^{\mathbb{Z}} \\ (x_n)_{n\in\mathbb{Z}} \mapsto (y_n)_{n\in\mathbb{Z}} \quad \text{where} \quad y_n = \frac{1}{2}\left(x_{n-1} + x_n\right). \end{cases}$$

Applying A is the same thing as convolution by the "gate function" $h = \frac{1}{2}\left(\delta_0 + \delta_1\right)$. It is not one-to-one, its kernel being (elementary verification)

$$\ker A = \{\lambda\left((-1)^n\right)_{n\in\mathbb{Z}}\,;\, \lambda \in \mathbb{R}\} = \{\lambda\left(\mathbf{1}_{2\mathbb{Z}} - \mathbf{1}_{2\mathbb{Z}+1}\right);\, \lambda\in\mathbb{R}\}.$$

This kernel expression holds too with the space $\ell^\infty(\mathbb{Z})$. But by restricting the linear transformation A to $c_0(\mathbb{Z})$ or to a smaller subspace, the kernel becomes $\{0\}$ and the map $x \mapsto A\,x$ is then one-to-one.

Here comes our main observations:

— $x = \frac{1}{2}\,\mathbf{1}_{\mathbb{Z}}$ is *perfect grey*;

— $x_n = 1$ if n is even, 0 otherwise is *macroscopically grey*;

— the same ones on, for example $\{0, \dots, 999\}$, will reveal to have quite different properties.

Pictures belonging to $\ell^\infty(\mathbb{Z})$. Let the convolution by h be the blurring action. Then $\mathbf{1}_{2\,\mathbb{Z}} * h$ and $\frac{1}{2}\,\mathbf{1}_{\mathbb{Z}}$ $(= [\frac{1}{2}\,\mathbf{1}_{\mathbb{Z}}] * h)$ are identical. Inversion of A and deconvolution are impossible.

Pictures belonging to $\mathbb{R}^{(\mathbb{Z})}$. Then A is one-to-one (its kernel, $\ker A$, vanishes). If $x \in \mathbb{R}^{(\mathbb{Z})}$ the blurred picture $h * x$ has also compact support and convolution with H defined in (6) is possible. Thus

$$(x * h) * H = x * (h * H) = x * \delta_0 = x\,. \tag{8}$$

But some different x can give very closed blurred pictures. Precisely take

$$x_n = \begin{cases} 1 & \text{if } n \text{ is even and } 0 \le n \le 998 \\ 0 & \text{otherwise} \end{cases}$$

(there are 500 pixels with value 1). The blurring gives the picture $y = h * x$ with

$$y_n = \frac{1}{2}\,x_n + \frac{1}{2}\,x_{n-1} = \frac{1}{2} \text{ for } 0 \le n \le 999 \text{ and } 0 \text{ otherwise} \tag{9}$$

(there are 1000 pixels with value 1/2).

But the almost perfect grey picture $\tilde{x} = \frac{1}{2}\,\mathbf{1}_{\{0,999\}}$ (it is grey on a large interval) is blurred into $\tilde{y}$ where

$$\tilde{y}_n = \begin{cases} \frac{1}{2} & \text{if } 1 \leq n \leq 999 \\ \frac{1}{4} & \text{if } n = 0 \text{ or } 1000 \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

which is very closed to y obtained in (9). This illustrates the ill-posedness of the inversion problem[7]. Note also that despite the possibility of deconvolution (8), H is an unbounded measure with unbounded support. This inversion is in some sense *academical*.

Practitioners use high-pass filters under the form of convolution with a small supported *mask* (look on the Net at "sharpening"), for example in dimension 2 a measure supported by $\{-1,0,1\} \times \{-1,0,1\}$ as maybe

0	-1	0
-1	5	-1
0	-1	0

or

-1	-1	-1
-1	9	-1
-1	-1	-1

the sum of all coefficients being 1.

5 Exercices.

When the picture x or $\tilde{x}$ belong to $\mathbb{R}^{(\mathbb{Z})}$, deconvolution works theoretically perfectly.

Case of macroscopic grey. As for $y = A(x)$ given in (9) the formula

$$\sum_{k \in \mathbb{Z}} y_k \, H_{n-k}$$

(H_m is the m-th term of H defined in (6)) gives exactly x_n. This could be an exercice. The inverse J_1 (cf. (5)) can equally do the job, with

$$\sum_{k \in \mathbb{Z}} y_k \, J_{1,n-k} \quad \text{where} \quad J_{1,m} = 2\,(-1)^m \text{ for } m \geq 0\,.$$

[7] In this example there is a bad behavior as analysed in *sampling theory*.

Case of almost perfect grey. As for $\tilde{y} = A(\tilde{x})$ given in (10) the formulas

$$\sum_{k \in \mathbb{Z}} \tilde{y}_k \, H_{n-k} \quad \text{or} \quad \sum_{k \in \mathbb{Z}} \tilde{y}_k \, J_{1,n-k}$$

give exactly $\tilde{x}_n$.

6 About the threshold $1/2$.

In [D] C. Duval studies convolution by $a\,\delta_0 + \alpha\, g(x)\,dx$ imposing $a > \frac{1}{2}$.

We refinded this in Lemma 1 where the multiplicative factor $-\dfrac{1-a}{a}$ has absolute value < 1 if and only if $a > \frac{1}{2}$.

We refinded again this in Theorem 2 where the multiplicative factor λ belongs to $]-1, 0[$ and badly tends to -1 when $a \searrow \frac{1}{2}$ (equivalently $b \nearrow \frac{1}{4}$).

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