Abstract

Blurring of a photographic image by a wrong focus can be modeled by convolution. This paper discusses some points for the inverse operation with particular interest on the set of integers \( \mathbb{Z} \).

**MSC2010:** 65R30 (Improperly posed problems), 94A08 (Image processing).

1 Introduction.

Briefly a problem is ill-posed if there is a “bad” transformation \( A \) (“signal” \( \mapsto \) “blurred signal” for example) and one tries recovering the preimage of any \( y \), expecting to find an \( x \) such that \( Ax = y \). Difficulties could lie in: \( A \) is not one-to-one, or very different initial points may have very closed image (see Section 4), or (frequently this happens simultaneously) the map \( A \) is not onto.

Photographic images often present blurring, for example due to a wrong focus setting. Several other defects due to different causes are possible (cf. [Be]). Defect of focus is roughly equivalent to convolution of the image source with the brightness of the image of one point light in 0\(^1\). Numerous papers use the word deconvolution. Is it more than a word? Surely this belongs to the class of ill-posed problems (see [TA, Ch.IV pp.91–115]).

Several authors add stochastic component. There is a clear reason: when the map “signal” \( \mapsto \) “blurred signal” is not onto (this may highly depends on the functional space under consideration), finding a preimage to any point

\[ \text{The density could be } k1_{B(0,r)} \text{ where } r > 0 \text{ is a radius and } k = (\pi r^2)^{-1}. \]

1 Question in dimension 2 and with the Euclidean norm: is it a zero divisor for convolution? We will see (Section 2) that in dimension 1 we do have a zero divisor.

1
in the target space of $A$ needs some stochastic adjustment. Many papers speak of \textit{bayesian}. The signal itself can be seen as a trajectory of a stochastic process\footnote{Cf. the Wiener filter, I learned in R. Pallu de La Barrière [PB].}. \textit{Robustness} may also be refered to because the perturbation is not exactly known. A lot of recent papers use \textit{wavelets}\footnote{Using the Fourier transform is tempting but disappoitting.}.

Literature is prolific and difficult to understand. The words \textit{mask}, \textit{sharpening} are keys on the Net, as also \textit{filtration} and \textit{denoising}. An astonishing algorithm is due to P.H. van Cittert: see Wikipedia (German) [J, VC]. A paper with an heralding title, which quotes van Cittert, and illustrate the interest to the question outside of the purely mathematical world is [Bi].

We will give some calculi with the space $\mathbb{Z}$ (dimension 1) and refind the threshold $\frac{1}{2}$ highlighted by C. Duval [D] (see Section 6).

I thank Manuel Monteiro Marques for his constant encouragements and Paul Raynaud de Fitte for his invaluable bibliographical helps.


Convolution and Fourier transform have as framework $\mathbb{R}^d$ (which is its own dual group), or $\mathbb{Z}^d$ and its dual group\footnote{Cf. Fourier series.} $\mathbb{T}^d$ (maybe the groups $\mathbb{Z}/n\mathbb{Z}$?). When $\mu$ and $\nu$ are bounded measures on $\mathbb{R}^d$, their convolution product, denoted by $\mu \ast \nu$, is the image (pushforward\footnote{Cf. the writing $\mu \ast \nu = S_\#(\mu \otimes \nu)$ where $S$ denotes the sum.}) by $(x, y) \mapsto x + y$ of their product $\mu \otimes \nu$ (cf. the sum of independant random variables in Probability). For Lebesgue integrable functions on $\mathbb{R}^d$, their convolution product is classically

$$ f \ast g : x \mapsto \int_{\mathbb{R}^d} f(x - y) g(y) \, dy. $$

The excellent paper by K.A. Ross [R] examines mainly convolution of $L^1$ functions.

Concerning convolution of distributions, L. Schwartz begins by the case of two when one of them has compact support. Then he proves [S, vol.2, Th.VII p.14] that the convolution of a finite number of distributions which have all, except one, compact supports is associative et commutative and then [S, vol.2, ch.6 §5 p.26] moves to more general situations. He proves that, in dimension $d = 1$, the \textit{convolution algebra} $\mathcal{D}'_+$ (the set of distributions with supports limited on left) has no zero divisors [S, vol.2, ch.6 Th.XIV p.29].
The same result holds for $D'_{\mathbb{R}}$ (the set of distributions with supports limited on right).

The unit mass in 0, $\delta_0$, is always a neutral element and convolution by $\delta_x$ ($x \in \mathbb{R}^d$) amounts to translation by the vector $x$. Uniqueness of a possible inverse holds when one works in a subspace where associativity holds (see (1) hereafter). We will see cases where several inverses do coexist (Theorem 1).

Let us show that $\delta_x$ has as unique inverse $\delta_{-x}$ (who doubts it?). Suppose $H$ is another distribution inverse of $\delta_x$. Among the three distributions $\delta_x$, $\delta_{-x}$ and $H$, two have compact support, hence associativity holds and

$$(1) \quad H = H * \delta_0 = H * (\delta_x * \delta_{-x}) = (H * \delta_x) * \delta_{-x} = \delta_0 * \delta_{-x} = \delta_{-x}.$$ 

When dealing with $\mathbb{Z}$ and measures such as $\mu = \sum_{n \in \mathbb{Z}} x_n \delta_n$, or $\nu = \sum_{n \in \mathbb{Z}} y_n \delta_n$, the point of view of convolution is to consider the function $n \mapsto x_n$ defined on $\mathbb{Z}$ (resp. $n \mapsto y_n$). The convolution of $\mu$ and $\nu$ returns to the convolution $z := x * y$ where $z_n = \sum_{k \in \mathbb{Z}} x_{n-k} y_k$. Next $h$ will equivalently denote a measure or a function on $\mathbb{Z}$.

**Examples of zero divisors.**

1) With $\mathbb{R}$ let consider the *gate function* $h = 1_{[-1,1)}$. Then

$$h * 1_{\cup_{n \in \mathbb{Z}} [2n,2n+1]} = h * \left( \frac{1}{2} 1_{\mathbb{R}} \right)$$

hence $f \mapsto h * f$ is not injective on $L^\infty(\mathbb{R})$, and one has a zero divisor:

$$h * \left( 1_{\cup_{n \in \mathbb{Z}} [2n,2n+1]} - \frac{1}{2} 1_{\mathbb{R}} \right) = 0.$$

2) With $\mathbb{Z}$, take $h := \frac{1}{2} \delta_0 + \frac{1}{2} \delta_1$ or $h := \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_0 + \frac{1}{4} \delta_1$. Then there holds

$$(2) \quad h * 1_{2\mathbb{Z}} = h * \left( \frac{1}{2} 1_{\mathbb{Z}} \right),$$

hence $f \mapsto h * f$ is not injective on $\ell^\infty(\mathbb{Z})$, and one has the zero divisor:

$$h * \left( 1_{2\mathbb{Z}} - \frac{1}{2} 1_{\mathbb{Z}} \right) = 0.$$
3 Convolution and inverse, particular cases.

Let us begin by

\[ h = a \delta_0 + (1 - a) \delta_1 \quad (a \in ]0, 1[) \] (a kind of “gate function”).

Lemma 1 Let \( a \in ]0, 1[ \) and \( h = a \delta_0 + (1 - a) \delta_1 \). Then an inverse of \( h \) in \( \mathcal{D}'_+ (\mathbb{R}) \) is

\[ J = \frac{1}{a} \delta_0 - \frac{1 - a}{a^2} \delta_1 + \frac{(1 - a)^2}{a^3} \delta_2 + \ldots \] (the limit is for the weak topology \( \sigma (\mathcal{D}', \mathcal{D}) \))

Proof. Indeed

\[
h \ast \frac{1}{a} \sum_{n=0}^{k} \left[ - \frac{1 - a}{a} \right]^n \delta_n = \delta_0 + (-1)^k \left[ 1 - \frac{1}{a} \right]^{k+1} \delta_{k+1} \rightarrow \delta_0
\]

because for any \( \alpha_n, \alpha_n \delta_n \rightarrow 0 \) in the topology \( \sigma (\mathcal{D}', \mathcal{D}) \) when \( n \rightarrow \infty \). □

Lemma 2 An inverse of \( h \) in \( \mathcal{D}'_+ (\mathbb{R}) \) is

\[ \frac{1}{1 - a} \delta_{-1} - \frac{a}{(1 - a)^2} \delta_{-2} + \frac{a^2}{(1 - a)^3} \delta_{-3} - \frac{a^3}{(1 - a)^4} \delta_{-4} + \ldots \] (the limit still for \( \sigma (\mathcal{D}', \mathcal{D}) \))

Proof. One can write

\[ h = (1 - a) \delta_1 \ast (\delta_0 + \frac{a}{1 - a} \delta_{-1}) \]

Then \( (1 - a) \delta_1 \) admits the inverse \( \frac{1}{1 - a} \delta_{-1} \) and for the second factor one can develop “on left” as in the preceding lemma. □

Theorem 1 The distribution \( \frac{1}{2} (\delta_0 + \delta_1) \) on \( \mathbb{R} \) admits several inverses in \( \mathcal{D}' \) with respect to convolution (the limits are for \( \sigma (\mathcal{D}', \mathcal{D}) \)):

\[ J_1 = 2 \lim_{k \rightarrow \infty} \sum_{n=0}^{k} (-1)^n \delta_n = 2 (\delta_0 - \delta_1 + \delta_2 - \delta_3 + \ldots) , \]

\[ ^6 \text{Cf. the known formula} \ (1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots \text{for} \ x \in \mathbb{R} .\]
\[ J_2 = 2 \lim_{k \to \infty} \sum_{n=1}^{k} (-1)^{n-1} \delta_{-n} = 2 (\delta_{-1} - \delta_{-2} + \delta_{-3} - \delta_{-4} + \ldots) \]

and specially \( H = \frac{1}{2} (J_1 + J_2) \) i.e.
\[ (J) \quad H = \ldots - \delta_{-4} + \delta_{-3} - \delta_{-2} + \delta_{-1} + \delta_0 - \delta_1 + \delta_2 - \delta_3 + \ldots \]

Moreover for any \( f \in \mathbb{R}^{(\mathbb{Z})} \) (the space of real sequences on \( \mathbb{Z} \) with compact supports)
\[ (f * h) * H = f. \]

**Remarks.** For any \( \lambda \in \mathbb{R} \), \( \lambda J_1 + (1 - \lambda) J_2 \) is also an inverse of \( h \). And \( J_1 - J_2 \) forms with \( h \) a couple of zero divisors.

**Proof.** The lemmas imply the assertions about inverses. The last formula follows from the fact that \( f \) and \( h \) have compact supports, hence \( (f * h) * H = f * (h * H) = f * \delta_0. \)

Now we turn to a measure carried by \( \{ -1, 0, 1 \} \), still positive with total mass 1. With the parameter \( a \in \left] \frac{1}{2}, 1 \right[ \)
\[ (7) \quad h := \frac{1-a}{2} \delta_{-1} + a \delta_0 + \frac{1-a}{2} \delta_1 \]

or, with the parameter \( b = \frac{1-a}{2} \in \left] 0, \frac{1}{4} \right[ \) which will be often better suited,
\[ h := b \delta_{-1} + (1 - 2b) \delta_0 + b \delta_1. \]

**Lemma 3** Let \( b \in \left] 0, \frac{1}{4} \right[ \). Then
\[ \lambda = \frac{1}{2b} \left[ 2b - 1 + \sqrt{1 - 4b} \right] \]
belongs to \( ]-1, 0[ \), tends to 0 if \( b \to 0 \), and tends to \(-1 \) if \( b \to 1/4 \).

**Proof.** Elementarily \( \lambda \) is a root of the equation \( \lambda^2 + \frac{1-2b}{b} \lambda + 1 = 0 \). One has \( \lambda \leq 0 \) because
\[ 2b - 1 + \sqrt{1 - 4b} \leq 0 \iff \sqrt{1 - 4b} \leq 1 - 2b \]
\[ \iff 1 - 4b \leq (1 - 2b)^2 \]
\[ \iff 1 - 4b \leq 1 - 4b + 4b^2 \]
and \( \lambda > -1 \) because
\[ 2b - 1 + \sqrt{1 - 4b} > -2b \iff \sqrt{1 - 4b} > 1 - 4b \]
which holds, since on \( ]0, 1[ \), \( \sqrt{x} \) is > \( x \). The convergences are easy. □
Theorem 2 Let \( h \) given by (7)

\[
h := \frac{1-a}{2} \delta_{-1} + a \delta_0 + \frac{1-a}{2} \delta_1.
\]

Let \( c \) defined by

\[
\forall n \in \mathbb{Z}, c_n = \lambda^{|n|} \left( \sqrt{1-4b} \right)^{-1}.
\]

The \( c_n \) are alternatively \( > 0 \) and \( < 0 \) and \( \sum_{n \in \mathbb{Z}} c_n = 1 \). The measure (or sequence) \( c \) is an inverse of \( h \), that is \( h \ast c = \delta_0 \). Moreover for any \( f \in \ell^\infty(\mathbb{Z}) \)

\[
(f \ast h) \ast c = f.
\]

Remark. Since \(-1 < \lambda < 0\), \( c \) considered as a function oscillates as the famous cardinal sine function: \( \text{sinc} x = \frac{\sin x}{x} \) (cf. also the mexican hat). This seems quite general. For another comment see Section 6.

Proof. 1) One has

\[
\sum_{n \in \mathbb{Z}} c_n = c_0 + 2 \sum_{n \geq 1} c_n
\]

\[
= c_0 \left( 1 + 2 \sum_{n \geq 1} \lambda^n \right)
\]

\[
= c_0 \left( 1 + 2 \frac{\lambda}{1-\lambda} \right)
\]

\[
= \frac{1}{\sqrt{1-4b}} \frac{1+\lambda}{1-\lambda}
\]

\[
= \frac{1}{\sqrt{1-4b}} \frac{4b - 1 + \sqrt{1-4b}}{1 - \sqrt{1-4b}}
\]

\[
= 1.
\]

2) Firstly

\[
(h \ast c)_n = \sum_{i \in \mathbb{Z}} h(n-i) c_i = \sum_{i \in \mathbb{Z}} h(i) c_{n-i}.
\]
For \( n = 0 \) this gives

\[
(h \ast c)_0 = h(-1) c_1 + h(0) c_0 + h(1) c_{-1}
\]

\[
= b \frac{\lambda}{\sqrt{1 - 4b}} + \frac{1}{\sqrt{1 - 4b}} + b \frac{\lambda}{\sqrt{1 - 4b}}
\]

\[
= \frac{1}{\sqrt{1 - 4b}} [2b \lambda + 1 - 2b]
\]

\[
= \frac{1}{\sqrt{1 - 4b}} \left[ 2b - 1 + \sqrt{1 - 4b} + 1 - 2b \right]
\]

\[
= 1.
\]

For \( n \geq 1 \) this gives

\[
(h \ast c)_0 = h(-1) c_{n+1} + h(0) c_n + h(1) c_{n-1}
\]

\[
= b (c_{n-1} + c_{n+1}) + (1 - 2b) c_n
\]

\[
= c_0 \left[ b \lambda^{n-1} + b \lambda^{n+1} + (1 - 2b) \lambda^n \right]
\]

\[
= \frac{\lambda^{n-1}}{\sqrt{1 - 4b}} \left[ b + (1 - 2b) \lambda + b \lambda^2 \right]
\]

\[
= 0.
\]

because \( \lambda^2 + \frac{1-2b}{b} \lambda + 1 = 0 \).

3) As for \((f \ast h) \ast c\), the functions are respectively, bounded for \( f \), with compact support for \( h \), integrable for \( c \) (convergent sum). So associativity holds. \( \square \)

4 Illustration (pictures on \( \mathbb{Z} \)).

A monochrome photographic image can be modeled by a (measurable) function \( f : \mathbb{R}^2 \rightarrow [0, 1] \), \( f \) measuring the brightness.

We will expose some examples with \( f : \mathbb{Z} \rightarrow [0, 1] \), that is a one-dimensional picture formed from pixels. So the basic space is \( \ell^\infty(\mathbb{Z}) \). Another natural space is \( \ell^p(\mathbb{Z}) \) that is the space of real sequences on \( \mathbb{Z} \) with compact supports (this is Bourbaki’s notation); it is a natural space since pictures do have compact supports. Other vector spaces could be considered in abstract studies (\( p \in ]1, \infty[ \)):

\[
\ell^1(\mathbb{Z}) \subset \ell^p(\mathbb{Z}) \subset c_0(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z}) \subset \mathbb{R}^\mathbb{Z}.
\]
As measures spaces, $\mathbb{R}^\mathbb{Z} \sim \mathcal{M}(\mathbb{Z})$ (the space of all measures on $\mathbb{Z}$) and $\ell^1(\mathbb{Z}) \sim \mathcal{M}^b(\mathbb{Z})$ (the space of all bounded measures on $\mathbb{Z}$).

Let us consider the linear map

$$A : \mathbb{R}^\mathbb{Z} \rightarrow \mathbb{R}^\mathbb{Z}$$

where

$$(x_n)_{n \in \mathbb{Z}} \mapsto (y_n)_{n \in \mathbb{Z}} \quad \text{where} \quad y_n = \frac{1}{2} (x_{n-1} + x_n).$$

Applying $A$ is the same thing as convolution by the “gate function” $h = \frac{1}{2} (\delta_0 + \delta_1)$. It is not one-to-one, its kernel being (elementary verification)

$$\ker A = \{ \lambda ((-1)^n)_{n \in \mathbb{Z}} : \lambda \in \mathbb{R} \} = \{ \lambda (1_{2\mathbb{Z}} - 1_{2\mathbb{Z}+1}) : \lambda \in \mathbb{R} \}.$$

This kernel expression holds too with the space $\ell^\infty(\mathbb{Z})$. But by restricting the linear transformation $A$ to $c_0(\mathbb{Z})$ or to a smaller subspace, the kernel becomes $\{0\}$ and the map $x \mapsto Ax$ is then one-to-one.

Here comes our main observations:

— $x = \frac{1}{2} 1_\mathbb{Z}$ is perfect grey;
— $x_n = 1$ if $n$ is even, 0 otherwise is macroscopically grey;
— the same ones on, for example $\{0, \ldots, 999\}$, will reveal to have quite different properties.

**Pictures belonging to $\ell^\infty(\mathbb{Z})$.** Let the convolution by $h$ be the blurring action. Then $1_{2\mathbb{Z}} * h$ and $\frac{1}{2} 1_\mathbb{Z}$ ($= [\frac{1}{2} 1_\mathbb{Z}] * h$) are identical. Inversion of $A$ and deconvolution are impossible.

**Pictures belonging to $\mathbb{R}(\mathbb{Z})$.** Then $A$ is one-to-one (its kernel, $\ker A$, vanishes). If $x \in \mathbb{R}(\mathbb{Z})$ the blurred picture $h * x$ has also compact support and convolution with $H$ defined in (6) is possible. Thus

$$(8) \quad (x * h) * H = x * (h * H) = x * \delta_0 = x.$$ 

But some different $x$ can give very close blurred pictures. Precisely take

$$x_n = \begin{cases} 1 & \text{if } n \text{ is even and } 0 \leq n \leq 998 \\ 0 & \text{otherwise} \end{cases}$$

(there are 500 pixels with value 1). The blurring gives the picture $y = h * x$ with

$$(9) \quad y_n = \frac{1}{2} x_n + \frac{1}{2} x_{n-1} = \frac{1}{2} \quad \text{for } 0 \leq n \leq 999 \text{ and } 0 \text{ otherwise}$$

(there are 1000 pixels with value $1/2$).
But the almost perfect grey picture $\tilde{x} = \frac{1}{2} \mathbf{1}_{(0,999)}$ (it is grey on a large interval) is blurred into $\tilde{y}$ where

\begin{equation}
\tilde{y}_n = \begin{cases}
\frac{1}{2} & \text{if } 1 \leq n \leq 999 \\
\frac{1}{4} & \text{if } n = 0 \text{ or } 1000 \\
0 & \text{otherwise}
\end{cases}
\end{equation}

which is very close to $y$ obtained in (9). This illustrates the ill-posedness of the inversion problem\(^7\). Note also that despite the possibility of deconvolution (8), $H$ is an unbounded measure with unbounded support. This inversion is in some sense *academic*.

Practitioners use high-pass filters under the form of convolution with a small supported *mask* (look on the Net at “sharpening”), for example in dimension 2 a measure supported by $\{-1,0,1\} \times \{-1,0,1\}$ as maybe

\[
\begin{array}{ccc}
0 & -1 & 0 \\
-1 & 5 & -1 \\
0 & -1 & 0
\end{array}
\quad \text{or} \quad
\begin{array}{ccc}
-1 & -1 & -1 \\
-1 & 9 & -1 \\
-1 & -1 & -1
\end{array}
\]

the sum of all coefficients being 1.

## 5 Exercices.

When the picture $x$ or $\tilde{x}$ belong to $\mathbb{R}^{(2)}$, deconvolution works theoretically perfectly.

**Case of macroscopic grey.** As for $y = A(x)$ given in (9) the formula

$$\sum_{k \in \mathbb{Z}} y_k H_{n-k}$$

($H_m$ is the $m$-th term of $H$ defined in (6)) gives exactly $x_n$. This could be an exercice. The inverse $J_1$ (cf. (5)) can equally do the job, with

$$\sum_{k \in \mathbb{Z}} y_k J_{1,n-k} \quad \text{where} \quad J_{1,m} = 2 (-1)^m \text{ for } m \geq 0.$$\(^7\) In this example there is a bad behavior as analysed in *sampling theory*.
Case of almost perfect grey. As for $\tilde{y} = A(\tilde{x})$ given in (10) the formulas

$$\sum_{k \in \mathbb{Z}} \tilde{y}_k H_{n-k} \text{ or } \sum_{k \in \mathbb{Z}} \tilde{y}_k J_{1,n-k}$$

give exactly $\tilde{x}_n$.

6 About the threshold $1/2$.

In [D] C. Duval studies convolution by $a \delta_0 + \alpha g(x) \, dx$ imposing $a > \frac{1}{2}$.

We refinded this in Lemma 1 where the multiplicative factor $\frac{1-a}{a}$ has absolute value $< 1$ if and only if $a > \frac{1}{2}$.

We refinded again this in Theorem 2 where the multiplicative factor $\lambda$ belongs to $]1,0[$ and badly tends to $-1$ when $a \downarrow \frac{1}{2}$ (equivalently $b \uparrow \frac{1}{2}$).

References

[Be] Bergounioux, M., Quelques méthodes mathématiques pour le traitement d’image, Cours de DEA, Université d’Orléans (2008) 110 pages. 
https://cel.archives-ouvertes.fr/cel-00125868v4/document

https://hal.archives-ouvertes.fr/jpa-00244050/document

https://hal.archives-ouvertes.fr/hal-01199599/document

https://de.wikipedia.org/wiki/Van-Cittert-Dekonvolution


