Young Measures
from Ghouila-Houri to Raynaud de Fitte

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1. In 1963 in Caen (a main town in Normandy, the other large town is Rouen), I was a beginner in a team whose chief was PALLU DE LA BARRIÈRE. There arrives in 1964 a young Professor Alain GHOUILA-HOURI: he died in 1966 at 27! He was very brilliant: he has already written a book with Claude BERGE: "Programmes, jeux, réseaux de transport" (this is Graph Theory). He gave lectures on the Brouwer fixed-point theorem with Sperner's lemma. He worked in optimal control theory, on problems without solutions and the way to get generalized solutions. Then he disappeared (after one universitary year?). Our chief Pallu refused relating what he knew about this. I learned far later that Ghouila-Houri committed suicide (see on the Net page 4 of the book by Ramírez Alfonsín and Reed, "Perfect Graphs").

Why optimal control? The first Spoutnik appeared in October 1957. The book "The mathematical theory of optimal processes" (in Russian language) by Pontryagin, Boltyanskii, Gamkrelidze and Mishchenko seemed containing the basic mathematical results for launching of space-rockets and satellites.

2. Consider a Calculus of Variations problem

$$\text{minimize } \int_0^1 L(x(t), \dot{x}(t))\, dt \tag{1}$$

where $x : [0, 1] \to \mathbb{R}$, $\dot{x} = \frac{dx}{dt}$, $x(0) = 0$,
or an Optimal Control problem

$$\dot{x}(t) = f(t, x(t), u(t))$$

$t \in [0, 1]$, $u(t)$ belonging to a set U ($u(t)$ is the value of the control at time t), $x(0)$ given, the criterion (to be minimized) being the value of a continuous function g at $x(1)$. In most cases there is not an optimal solution[1]. For example with

$$L(p,q) = p^2 + (q^2-1)^2\,,$$

the infimum in (1) is 0 but cannot be reached. What is a generalized solution? Consider at time t the "generalized velocity"

$$\nu_t := \frac{1}{2}\big(\delta_{-1} + \delta_1\big) \tag{2}$$

which is a probability on $\mathbb{R}$:
— as for the velocity it is the barycenter of ν_t, that is 0;
— as for the criterion, the action of ν_t is $\int_{\mathbb{R}} L(x(t),q)\,d\nu_t(q) = 0$ (for $x(t) = 0$).
So we have got an optimal generalized solution.

A minimizing sequence for (1) is the sequence $(x_n)_n$ where x_n is the primitive null at $t=0$ of the Rademacher function r_n. Let us recall that r_n switches between $+1$ and -1: $r_n(t) = 1$ on each $[k2^{-n},(k+1)2^{-n}]$ for k even and $r_n(t) = -1$ on the remaining of $[0,1]$. To r_n is associated, at time t, the measure $\mu_{n,t} = \delta_{r_n(t)}$. In some sense the family $(\nu_t)_t$ is the limit of the sequence (the index is n) of families $(\mu_{n,t})_t$.

In my opinion the convergence of μ_n (associated to r_n) to ν defined by (2) is a visual phenomenon. More generally the n-th function may have non periodical oscillations; a Young measure will appear when oscillations increase with n.

3. We must precise several points:

1. the two presentations: the disintegrated one and the global one;
2. the embedding of functions in the space of Young measures;
3. the convergence or the topology on the space of Young measures;
4. the distinction between the case where all values of velocities or control values are constrained in a compact and the general case ($q \in \mathbb{R}$ for example). This is where Ghouila-Houri work has a limited field. But his paper was modern, very precise, complete and a source for several successors (Ekeland, Berliocchi, Lasry).

[1] In the old times Calculus of Variations was on the "smooth side" and the new Optimal Control Theory appeared on the "non smooth side". As soon as Nonsmooth Analysis developed the differences go decreasing. For hypotheses allowing transformation of a problem of one type in a problem of the other type see some results in Clarke's book "Optimization and Nonsmooth Analysis" (1983), Section 5.4 p.219, and see references in the note p.289.

1. *Disintegration.* A measure ν on $[0,1]\times\mathbb{R}$ whose projection on $[0,1]$ is the Lebesgue measure disintegrates in a family $(\nu_t)_t$ of probabilities on $\mathbb{R}$. This is one-to-one up to equality a.e. of families. One can handle this by the formula: for any function $\theta \geq 0$ measurable on $[0,1]\times\mathbb{R}$ (such a function is called an integrand)

$$\iint_{[0,1]\times\mathbb{R}} \theta \, d\nu = \int_{[0,1]} \Big[\int_{\mathbb{R}} \theta(t,q) \ d\nu_t(q) \Big] \, dt$$

2. *Embedding.* To a function r (I use this letter because of the r_n above) is associated $\mu_t = \delta_{r(t)}$. It amounts for μ to be the push-forward of the Lebesgue measure by $t \mapsto (t, r(t))$. Note that the mass is on the graph of r, but in accordance to the abscissa (proportionally to the length of the curve would be meaningless for measurable functions). Alibert and Bouchitté explored the proportionality to $1+|r(t)|$ times the abscissa in order to encompass concentration.

3. *Topology.* The convergence is the "weak" convergence of measures:

$$\nu_n \to \nu \iff \iint_{[0,1]\times\mathbb{R}} \theta \, d\nu_n \to \iint_{[0,1]\times\mathbb{R}} \theta \, d\nu$$

for all θ in a class of test functions. In the non-compact case several spaces of test functions are possible (for example Carathéodory functions, measurable in t, continuous in q). My advice is that a small one makes easier getting topological results and then later one can prove that bigger spaces are allowed. The embedding of 2 permits to avoid the formulation of convergence of functions to a different object, as in several Tartar's papers. There one finds the following: up to extraction of a subsequence there exists a family $(\nu_t)_t$ such that for any ψ continuous on $\mathbb{R}$ (we assume that all values are constrained in a compact subset of $\mathbb{R}$)

$$\psi \circ r_n \rightharpoonup [t \mapsto \int_{\mathbb{R}} \psi \, d\nu_t] \, .$$

This means that we have the weak convergence in L^1:

$$\forall \varphi \in L^\infty([0,1]), \quad \int_0^1 \varphi(t)\, \psi(r_n(t))\, dt \to \int_0^1 \varphi(t) \Big[\int_{\mathbb{R}} \psi(q)\, d\nu_t(q) \Big] \, dt \, .$$

In terms of μ_n

$$\iint_{[0,1]\times\mathbb{R}} \varphi(t)\, \psi(q)\, d\mu_n(t,q) \to \iint_{[0,1]\times\mathbb{R}} \varphi(t)\, \psi(q)\, d\nu(t,q) \, .$$

Note that $(t,q) \mapsto \varphi(t)\,\psi(q)$ is a particular Carathéodory integrand.

Note that choosing a good space of test functions it is easy to check that when u_1 is 1-periodic on $\mathbb{R}$ the Young measures associated to $u_n := u_1(n\,.)$ converge to $\nu = dt \otimes \lambda$ where λ is the push forward of the Lebesgue on $[0,1]$ by u_1.

4. *Non compact space for q.* When the space of velocities or control values is not compact one has to assume tightness (Prohorov's condition ensuring no escape of mass to infinity). And, in all cases, for lower semi-continuity results with integrands, one has to control the negative parts if there are not 0, for example assuming uniform integrability.

4. The weak-strong lower semi-continuity result. If $x_n \to x$ strongly in L^1 and the Young measures associated to the y_n converge to ν then, for $\theta \geq 0$ on $[0,1] \times \mathbb{R}^2$ lower semi-continuous in the couple of variables (p,q),

$$\iint_{[0,1]\times\mathbb{R}} \theta(t, x(t), q)\, dt\, d\nu(t) \leq \liminf_{n\to\infty} \int_0^1 \theta(t, x_n(t), y_n(t))\, dt\,. \tag{3}$$

This remains valid with negative parts uniformly integrable (in short UI).

It applies to the Calculus of Variations when the unknown functions are one-dimensional valued (gradients are vectors). Then the Jensen's inequality works. Here are some details: let $x_n \to x$ strongly in L^1 and $y_n \rightharpoonup y$ weakly in L^1. Then up to a subsequence there exists a Young measure ν such that the Young measures associated to y_n converge to ν, $y(t)$ is the barycenter of ν_t and thanks to (3), if θ is as above, if $\theta(t,p,.)$ is convex and the negative parts are UI

$$\int_0^1 \theta(t, x(t), y(t))\, dt \leq \liminf_{n\to\infty} \int_0^1 \theta(t, x_n(t), y_n(t))\, dt\,.$$

This weak-strong lower semi-continuity result implies existence results in Calculus of Variations. In multidimensional problems (as Elasticity) the gradients are matrices and the open problem during several years was to get an inequality "à la Jensen" for the Young measures obtained as limits of gradients and quasi-convexity in the sense of Morrey. This was solved by Kinderlehrer and Pedregal in 1991.

5. In the bibliography I mention two-scale convergence which brings a second order analysis of oscillations, and some works on bounded sequences in L^1. Note that E.J. Balder proved many results using the Komlós theorem (which says that from any bounded sequence in L^1 one can extract a sequence verifying the strong law of large numbers!). I mention also existence without convexity: it appears in linear problems thanks to the Liapounov theorem and maybe in some other cases (Arrigo CELLINA is a specialist).

6. Paul RAYNAUD DE FITTE (a colleague in Rouen) began around 2000 a colossal work (he invited Castaing and me to join him). He discovered Topsøe results on tightness in spaces with very few continuous functions (the narrow topology has to receive a new definition). And he discovered the works of Rényi (the first in 1958) on "stable convergence" which is very closed to convergence of Young measures.

Some historical papers (chronological order)

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[Y3] Young, L.C., *Lectures on the calculus of variations and optimal control theory*, Foreword by Wendell H. Fleming, W.B. Saunders Co., Philadelphia-London-Toronto, Ont., 1969.

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Courses

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[Va1] Valadier, M., *Young measures*, Methods of nonconvex analysis (Varenna, 1989), 152–188, Lecture Notes in Math. **1446**, Springer, Berlin, 1990.

[Va2] Valadier, M., *A course on Young measures*, Workshop on Measure Theory and Real Analysis (Grado, 1993). Rend. Istit. Mat. Univ. Trieste **26** (1994), suppl., 349–394 (1995).

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[KP] Kinderlehrer, D. & Pedregal, P., *Characterizations of Young measures generated by gradients*, Arch. Rational Mech. Anal. **115** (1991), 329–365.

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[PV] Piccinini, L. & Valadier, M., *Uniform integrability and Young measures*, J. Math. Anal. Appl. **195** (1995), 428–439.

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[Va5] Valadier, M., *Young measures, weak and strong convergence and the Visintin-Balder theorem*, Set convergence in nonlinear analysis and optimization, Set-Valued Anal. **2** (1994), 357–367.

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[Va7] Valadier, M., *Analysis of the asymptotic distance between oscillating functions and their weak limit in L^2*, Advances in mathematical economics, **1** (1999), 99–113.

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Intentionally forgotten (invited in Nîmes)

Nguetseng, G.

Sychev, M.A.

Complementary list

Balder, E.J. (numerous papers; he brought most of the new ideas since 1979: see references in [Ba1, Ba2])

Berliocchi, H. & Lasry, J.-M. (paper in 1973)

Buttazzo, G. (book)

Dacorogna, B. (two books)

Murat, F. & Tartar, L. (several papers)

Pedregal, P. (book)

Roubíček, T. (book)